Coherent States in String Theory

G14GPD

MSc Dissertation in

Gravity, Particles and Fields

2015 - 2016

School of Mathematical Sciences

University of Nottingham

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Abstract

Within this dissertation , a number of theories have been discussed such as the 2-D conformal field theory in addition to provided an introduction to (bosonic string) string theory which is underpinned by the vertex operator and DDF operators which were used to create the coherent states. Pre-requisites for creating coherent states such as Nambu-Goto action, Polyakov action, Virasoro algebra, vertex operator, DelGuidice, DiVecchia and Fubini (DDF) operator have been analysed within the dissertation.

Acknowledgements

I would like to express my gratitude towards my supervisor Dr.Paul Saffin, whose help and guidance is the reason I have been able to learn and write this dissertation, also my personal tutor Dr. Madlin Guta for his support during my course period. The School of Mathematical Sciences and School of Physics. Also I would like to thank my friends Shailesh Dhasmana, Oliver Maddox, Ross Wakefield, Kulbir Thakur for a thoughtful and insightful discussion on my work. Last but not the least I would like to thank my parents without whom I would have never been able to be here, it was their blessings and support that I am here today and in the end I would like to thank the University of Nottingham, UK who considered me to be eligible for studying at such a prestigious institute.

Contents

1 Introduction

String theory is an aspiring and motivating concept, it takes us to the path of unifying all the fundamental forces into a simple quantum mechanical equation. In string theory the matter is not considered as a point particles, rather a tiny loopy string. Till today there has been no experimental evidence for the string theory.

String theory is a theory of quantum gravity [2]. It unified the Einstein's theory of general relativity with the quantum mechanics. There are several puzzles in the quantum gravity theory such as what actually happened at the beginning of time? Are singularities arising in black holes the end of time? What is the resolution of Information paradox?

The intention of this dissertation is to, create coherent states or Glauber Coherent States for cosmic strings. The reason we do this is because as cosmic strings are macroscopic and massive in nature and hence it gives us an evident idea, to have a classical interpretation towards them. They way to create these states goes with the understanding of the DDF operators, which are later operated on the physical states which in product gives us the coherent states.

In the discussion made within the section classical conformal invariance 2.1 we have seen how the conformal transformation takes place in the d-dimensions and the conformal groups properties. Which later on drove us to the Ein Einbin 2.2.1 where we saw how the world-line works for the mass-less particles and its action under reparametrisation. This path led us to understand the Nambu-Goto action for a relativistic action 2.26 which eventually took us to towards deriving and understanding the Polyakov action which is basically the classical a classical reformulation of the string action. We introduced the term Holomorphic and Anti-holomorphic in section 2.2.5,the holomorphy of the conformal transformation 2.2.5. Coming to the quantum aspects such as Operator Product Expansion (OPE) which has been discussed in section 2.3, following the understanding of OPE it leads to primary operators (section 3.1) and Virasoro algebra (section 3.1.1),this leads to the vertex operators which is one of the important concepts to be known within the dissertation.

Within section 3 a number of aspects have been analysed these include coherent states 3.2, coherent states in quantum mechanics (3.2.1) and coherent states in string theory (section 3.3). To create coherent states one needs to understand DDF operators 3.3.2 which consists of general equation of DDF on vacuum.

2 Background

In order to proceed towards the discussion, the concept of conformal field theory and string theory are vital to understand. In this dissertation there will be the use of holomorphy, complex analysis, 2D Euclidean world sheet which needs a background of contour integration, a brief discussion on the contour integration has been conducted in Appendix 5. The complete work will be done under the consideration of flat space, i.e., the coordinate will be selected according to the $g_{\mu\nu} = \eta_{\mu\nu}$, where the latter has the form $(-1, 1, 1, 1, \cdots)$ (which is the spacetime, not world sheet metric) and has only two eigenvalue -1 and $+1$. The convention used in my dissertation is -1 is the direction of time. The complete chapter is referenced from [1] and [4]

2.1 Classical Conformal Invariance

In mathematics, if the angle between two curves is invariant under the transformation, that can be called Conformal Invariance. The physical interpretation can be scripted as, if we consider a cube of dimension *'a'* is stretched to form an another cube of dimension $a + \delta$. Now in this transformation only the dimension of edges has been changed and not the angles, such a transformation can be termed as Conformal Invariance. The energy momentum tensor can be written as it's variation of action with respect to the metric,

$$\delta S = \frac{1}{2} \int d^d x \sqrt{g}\, T^{\mu\nu}\, \delta g_{\mu\nu} \tag{2.1}$$

As $\partial_\nu T^{\mu\nu} = 0$ because its a flat space. This tensor is defined as change of action S under space-time metric.

$$g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$$

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu} \tag{2.2}$$

General coordinate invariance is not of our interest. Symmetry which consist of metric and energy momentum tensor needs to be formulated. Such kind of symmetries are called Weyl invariance. By replacing $\delta g_{\mu\nu} = \omega(x) g_{\mu\nu}(x)$ into 2.1, we find

$$\delta S = \frac{1}{2} \int d^d x \sqrt{g}\, T^{\mu}_{\mu} \omega(x) \tag{2.3}$$

As the above equation 2.3 is true for any arbitrary ω. Hence, we conclude the condition for Weyl invariance is

$$T^{\mu}_{\mu} = 0 \tag{2.4}$$

The conformal transformation can be defined as a coordinate transformation which acts on a metric and also on a Weyl transformation [1]. Consider an action having a form

$$S = \int d^d x \mathcal{L}(\partial_x, g'_{\mu\nu}(x'), \phi(x)) \tag{2.5}$$

ϕ denotes any field that appears.

$$S = S' \equiv \int d^d x' \mathcal{L}(\partial'_x, g'_{\mu\nu}(x'), \phi'(x')) \tag{2.6}$$

$$g_{\mu\nu} \to g'_{\mu\nu} = \frac{\partial f^{\rho}}{\partial x'^{\mu}} \frac{\partial f^{\sigma}}{\partial x'^{\nu}} g_{\rho\sigma}(f(x'))$$

and transformation of a field ϕ depends on its spin. If it is a tensor of rank n,

$$\phi'_{\mu_1,\cdots,\mu_n}(x') = \frac{\partial f^{\nu_1}}{\partial x'^{\mu_1}} \cdots \frac{\partial f^{\nu_n}}{\partial x'^{\mu_n}} \phi'_{\nu_1,\cdots,\mu_n}(f(x'))) \tag{2.7}$$

for a scalar function $\phi(x)$ we find, $\phi'(x) = \phi(f(x'))$ and derivative of this function is,

$$\frac{\partial}{\partial x^{\mu}} \phi(x) \to \frac{\partial}{\partial x'^{\mu}} \phi'(x') = \frac{\partial}{\partial x'^{\mu}} \phi(f(x')) = \frac{\partial f^{\nu}}{\partial x'^{\mu}} \frac{\partial}{\partial f^{\nu}} \phi(f(x')) \tag{2.8}$$

as the transformation is like a vector. now Weyl invariance action is used to change the metric to the original form. Hence, we have;

$$S = S'' \equiv \int d^d x' \mathcal{L}(\partial x', g_{\mu\nu}(f(x')) = \int d^d x' \mathcal{L}(\partial x', g_{\mu\nu}(x), \phi'(x') \tag{2.9}$$

This is the conformal symmetry of the action. If we start from the flat space-time the metric remains unchanged. Certain fields those transforms as equation 2.7 under the conformal transformations are normally referred as *"Conformal fields"* or also *"Primary fields"*.

2.1.1 Conformal transformation in d-dimension

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x)$$

$$\begin{aligned} \frac{\partial x^{\rho}}{\partial x'^{\mu}} &= \frac{\partial}{\partial x'^{\mu}}(x'^{\rho} - \epsilon^{\rho}(x)) \\ &= \frac{\partial x'^{\rho}}{\partial x'^{\mu}} - \frac{\partial \epsilon^{\rho}(x)}{\partial x'^{\mu}} \end{aligned}$$

$$\frac{\partial x^{\rho}}{\partial x'^{\mu}} = \delta^{\rho}_{\mu} - \partial_{\mu}\epsilon^{\rho} \tag{2.10}$$

and

$$\delta g_{\mu\nu} = -\partial_{\mu}\epsilon_{\nu} - \partial_{\nu}\epsilon_{\mu} \tag{2.11}$$

Taking the trace of equation 2.11 we get,

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = g_{\mu\nu}\frac{2}{d}\partial \cdot \epsilon \qquad \cdots [g^{\mu\nu}g_{\mu\nu} = d] \tag{2.12}$$

Using $\partial^{\mu}\partial^{\nu}$ we can contract both the sides and the equation 2.12 will contract as,

$$\Box(\partial^\nu \cdot \epsilon_\nu) + \Box(\partial^\mu \cdot \epsilon_\mu) = \frac{2}{d}\partial^\mu\partial^\nu(g_{\mu\nu}(\partial\cdot\epsilon))$$
$$\implies \Box(\partial\cdot\epsilon) + \Box(\partial\cdot\epsilon) = \frac{2}{d}\partial^\mu(\partial^\nu g_{\mu\nu}(\partial\epsilon) + \partial^\nu(\partial\epsilon)g_{\mu\nu})$$

$$\left(1 - \frac{1}{d}\right)\Box\partial\epsilon = 0 \tag{2.13}$$

for d=1,

$$\Box\partial\epsilon = 0 \tag{2.14}$$

Now, contracting equations 2.13 and 2.14

$$\partial_\rho\partial^\nu\left[\left(1 - \frac{1}{d}\right)\Box\partial\epsilon\right] = 0 \tag{2.15}$$

$$\partial_\rho\left[\left(1 - \frac{1}{d}\right)\cdot\Box\partial\epsilon + \partial^\nu(\Box\partial\epsilon)\left(1 - \frac{1}{d}\right)\right] + \partial^\nu\left[\partial_\rho\left(1 - \frac{1}{d}\right)(\Box\partial\epsilon) + \partial_\rho(\Box\partial\epsilon)\left(1 - \frac{1}{d}\right)\right] \tag{2.16}$$

Equation 2.12 is contracted with $\partial_\rho\partial^\nu$ will give,

$$\Box\partial_\rho\epsilon_\mu + \left(1 - \frac{2}{d}\right)\partial_\rho\partial_\mu\partial\cdot\epsilon = 0$$

By adding the same equation with ρ and μ interchanged, we use equation 2.12 and 2.13 This results in,

$$\left(1 - \frac{2}{d}\right)\partial_\rho\partial_\mu\partial\cdot\epsilon = 0$$

Now, $\partial_\rho\partial_\mu\partial\cdot\epsilon = 0$ if $d > 2$.

Define $\Delta_{\rho\sigma\mu\nu} = \partial_\rho\partial_\mu\partial\cdot\epsilon \rightarrow$ this function is manifestly symmetric in the first 3 indices. This equation later on can be used to prove that a tensor with these symmetries must vanish.

Conformal group

Consider all the such Weyl transformation and create a set of it. This set turns out to be a group. Now, by the symmetry theorem which says, the symmetry depends on particular parameters, collection of such symmetries constitute a group. Such a group can be called a Conformal group.

As mentioned in the [1]

- Translations: $x^\mu \to x^\mu + \alpha^\mu$
- (Lorentz) Rotation: $x^\mu \to x^\mu + \omega^\mu_\nu x^\nu$
- Scale transformation: $x^\mu \to x^\mu + \sigma x^\mu$
- Special conformal transformation: $x^\mu \to x^\mu + b^\mu x^2 - 2x^\mu bx$

The corresponding generators which generates the above transformation is given by [1]

$$P_\mu = i\partial_\mu \tag{2.17}$$

$$M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \tag{2.18}$$

$$D = -ix^\mu \tag{2.19}$$

$$K_\mu = i(x^2\partial_\mu - 2x_\mu x^\nu)\partial_\nu \tag{2.20}$$

If we commute the operators P, M, D and K it is found that there is a formation of a closed algebra which is isomorphic to $SO(p+1, q+1)$.

2.2 String Theory

The content of this section 2.2 is been referred from the *String Theory by David Tong* [2] and *String Theory: Volume 1, An Introduction to the Bosonic String by Joseph Polchinski* [7]. First we introduce the Ein Einbin because it is the most important action which will lead us to the Polyakov action.

The fundamental change in the view point from Quantum field theory to string theory is that, at very high energy scale the fundamental particles does not behave as point particle, but as a 1-Dimensional stretched string.

2.2.1 Ein Einbin

Let x_μ be the coordinate of the string mass m. The action of such a string is given by,

$$S = \frac{1}{2}\int d\tau (e^{-1}\dot{x} - em^2) \tag{2.21}$$

$$\dot{x^2} = \dot{x^\mu}\dot{x^\nu}\eta_{\mu\nu}$$

The world line is,

$$S = \frac{1}{2}\int d\tau \;\; e(e^{-2}\dot{x^2} - m^2)$$
$$S = \frac{1}{2}\int d\tau \;\; e([e^2]^{-1}\dot{x^2} - m^2)$$

$$e = \sqrt{-g_{\tau\tau}} \quad ; \quad g_{\tau\tau}^{-1} = g^{\tau\tau}$$

$$S = \frac{1}{2}\int d\tau\sqrt{-g_{\tau\tau}}\;(g^{\tau\tau}\dot{x}^2 - m^2) \tag{2.22}$$

$$x^2 + e^2m^2 = 0 \tag{2.23}$$

$$\therefore \dot{x}^2 = -e^2m^2$$

Substitute 2.23 in 2.21

$$S = \frac{1}{2}\int d\tau \ \ (-e^{-1}e^2m^2 - em^2)$$
$$S = \frac{1}{2}\int d\tau \ \ (-em^2 - em^2) = 0$$
$$S = 0$$

Equation 2.22 works for massless particles ($m = 0$) Under the action of reparametrisation the action 2.22 remains invariant, which are written in the form similar to GR for transformation paramterisation by an infinitesimal we have,

$$\tau \to \tilde{\tau} = \tau - \eta(\tau)$$

$$\delta e = \frac{d}{d\tau}(\eta(\tau)e) \quad ; \quad \delta x^\mu = \frac{dx^\mu}{d\tau}\eta(\tau)$$

The transformation of Einbin e gives as the density on the worldline, which happens when each of the coordinates x^μ has a transformation as a worldline scalar.

2.2.2 The Nambu-Goto Action

Nambu-Goto action is a most elementary action in the bosonic string theory. It consider the beginning as an infinity thin string behavior, under the influence of Lagrangian principles. As the free point particle action is proportional to its length of its world-line , which is its proper time. Similarly the relativistic string action is proportional to the area of the traces by the string when it travels in spacetime. [13]

- Timelike coordinate - τ
- Spacelike coordinate - σ

Consider closed string and take σ as periodic with range,

$$\sigma \in [0, 2\pi)$$

$$\sigma^\alpha = (\tau, \sigma), \alpha = 0, 1$$

$$\gamma_{\alpha\beta} = \frac{\partial x^\mu}{\partial \sigma^\alpha} \frac{\partial x^\nu}{\partial \sigma^\beta} \eta_{\mu\nu} \tag{2.24}$$

Action proportional to the area of the worldsheet is

$$S = -T \int d^2\sigma \sqrt{-det\gamma} \tag{2.25}$$

T is proportionality constant (tension on the string, mass/unit length)

The pull back of the metric is defined by,

$$\gamma_{\alpha\beta} = \begin{pmatrix} \dot{x}^2 & \dot{x}x' \\ \dot{x}x' & x'^2 \end{pmatrix}$$

$\dot{x}^\mu = \frac{\partial x^\mu}{\partial \tau}$ and $x'^\mu = \frac{\partial x^\mu}{\partial \sigma}$

$$\begin{aligned} \therefore det\gamma &= \dot{x}^2 x'^2 - (\dot{x}x')(\dot{x}x') \\ &= \dot{x}^2 x'^2 - (\dot{x}x')^2 \end{aligned}$$

Hence, the action recasts into,

$$S = T \int d^2\sigma \sqrt{-\dot{x}^2 x'^2 - (\dot{x}x')^2} \tag{2.26}$$

This above equation 2.26 is the representation of Nambu-Goto action for a relativistic action. where the physical interpretation of action can be referred to as the area.

2.2.3 Polyakov Action

Polyakov action is 2-Dimensional action in Conformal Filed Theory which, describes the area covered by string in spacetime in string theory. [14] A string action which is classically equivalent to the Nambu-Goto action is, [2]

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-g}\ g^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta \eta_{\mu\nu} \tag{2.27}$$

$g = detg$ and $g^{\alpha\beta}$ is new introduced field.

The equation of motion can be described as,

$$x\partial_\alpha(-\sqrt{g}\ g^{\alpha\beta}\partial_\beta x^\mu) = 0 \tag{2.28}$$

which is similar to Nambu-Goto action. $g_{\alpha\beta}$ is independent variable, using

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\ g_{\alpha\beta}\delta g^{\alpha\beta} = +\frac{1}{2}\sqrt{-g}\ g^{\alpha\beta}\delta g_{\alpha\beta} \tag{2.29}$$

Modify equation 2.27 as,

$$S = -T \int d^2\sigma \delta(\sqrt{-g}\ g^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta \eta_{\mu\nu}) \tag{2.30}$$

$$\begin{aligned}
\delta S &= -T \int d^2\sigma \delta\sqrt{-g}(g_{\alpha\beta}\partial_\alpha x^\mu \partial_\beta \eta_{\mu\nu}) \\
&= -T \int d^2\sigma \left(\frac{1}{2}\sqrt{-g}\ g^{\alpha\beta}\delta g_{\alpha\beta}\right)(g_{\alpha\beta}\partial_\alpha x^\mu \partial_\beta \eta_{\mu\nu}) \\
&= -\frac{T}{2} \int d^2\sigma \delta g^{\alpha\beta} \left(\sqrt{-g}\ \partial_\alpha x^\mu \partial_\beta x^\nu - \frac{1}{2}\sqrt{-g}\ g_{\alpha\beta} g^{\rho\sigma}\partial_\rho x^\mu \partial_\sigma x^\nu\right)\eta_{\mu\nu} = 0
\end{aligned}$$

Which implies that the metric S or Polyakov action is symmetric.

2.2.4 Symmetries of the Polyakov action

- Poincaré invariance: Global symmetry on the worldsheet.

$$x^\mu \to \Lambda^\mu_\nu x^\nu + c^\mu$$

- Reparametrisation invariance: Gauge symmetry on the world-sheet. x^μ transforms as world-sheet and $g_{\alpha\beta}$ transforms according to 2d- metric,

$$x^\mu(\sigma) \to \tilde{x}^\mu(\tilde{\sigma}) = x^\mu(\sigma)$$

$$g_{\alpha\beta}(\sigma) \to \tilde{g}_{\alpha\beta}(\tilde{\sigma}) = \frac{\partial \alpha^\gamma}{\partial \tilde{\sigma}^\alpha}\frac{\partial \sigma^\delta}{\partial \tilde{\sigma}^\beta} g_{\gamma\delta}$$

By changing the coordinates,

$$\sigma^\alpha \to \tilde{\sigma}^\alpha = \sigma^\alpha - \eta^\alpha(\sigma) \qquad \cdots \forall\ \eta$$

Field transforms as,

$$\left.\begin{aligned} \delta x^\mu(\sigma) &= \eta^\alpha \partial_\alpha x^\mu \\ \delta g_{\alpha\beta}(\sigma) &= \nabla_\alpha \eta_\beta + \nabla_\beta \eta_\alpha \end{aligned}\right\} \tag{2.31}$$

and

$$\Delta_\alpha \eta_\beta = \partial_\alpha \eta_\beta + \Gamma^\sigma_{\alpha\beta}\eta_\sigma \tag{2.32}$$

In equation 2.32 $\Gamma^\sigma_{\alpha\beta}$ is a Levi-Civita connection which is,

$$\Gamma^\sigma_{\alpha\beta} = \frac{1}{2} g^{\sigma\beta}(\partial_\alpha g_{\beta\rho} + \partial_\beta g_{\beta\alpha} - \partial_\rho g_{\alpha\beta})$$

- Weyl Invariance:

$$x^\mu(\sigma) \to x^\mu(\sigma)$$

metric changes as,

$$g_{\alpha\beta}(\sigma)\Omega^2(\sigma)g_{\alpha\beta}(\sigma) \tag{2.33}$$

and

$$\Omega^2(\sigma) \to e^{2\phi(\sigma)} \tag{2.34}$$

$$\therefore \delta g_{\alpha\beta}(\sigma) = \delta(e^{2\phi(\sigma)}g_{\alpha\beta}(\sigma))$$

$$\delta g_{\alpha\beta}(\sigma) = 2\phi(\sigma)g_{\alpha\beta}(\sigma) \tag{2.35}$$

Now,

$$g_{\alpha\beta} = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$$

$$det\, g = \lambda_1\lambda_2\cdots\lambda_n$$

$$\delta det\, g = \left(\Sigma\frac{\delta\lambda_i}{\lambda_i}det\, g\right)$$

$$-\delta dig\, g = -(g^{\alpha\beta}\delta g_{\alpha\beta})det\, g$$

$$\begin{aligned}\delta\sqrt{-det\, g} &= \delta[-det\, g]^{\frac{1}{2}} = \frac{1}{2}[-det\, g]^{\frac{1}{2}}(\delta det\, g)\\ &= \frac{1}{2}[-det\, g]^{\frac{1}{2}}[-detg\, g^{\alpha\beta}\delta g_{\alpha\beta}]\\ &= \frac{1}{2}\sqrt{-det\, g}\, g^{\alpha\beta}\delta g_{\alpha\beta}\end{aligned}$$

and

$$g_{\alpha\beta} + \delta g_{\alpha\beta} =$$

$$g_{\alpha\beta} + \delta g_{\alpha\beta} = \begin{pmatrix} \lambda_1 + \delta\lambda_1 & & & \\ & \lambda_2 + \delta\lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n + \delta\lambda_n \end{pmatrix}$$

and

$$g^{\alpha\beta} = diag\left(\frac{1}{\lambda_1}\frac{1}{\lambda_2},\cdots\right)$$

also,

$$\delta g_{\alpha\beta} = \begin{pmatrix} \delta\lambda_1 & & \\ & \delta\lambda_2 & \\ & & \ddots \end{pmatrix}$$

$$\begin{aligned} det(g_{\alpha\beta} + \delta g_{\alpha\beta}) &= (\lambda_1 + \delta\lambda_1)(\lambda_2 + \delta\lambda_2)\cdots \\ &= \lambda_1\lambda_2\cdots + \delta\lambda_1\lambda_2\cdots + \lambda_1\delta\lambda_2\lambda_3\cdots \\ &= det\, g + \left(\frac{\delta\lambda_1}{\lambda_1} + \frac{\delta\lambda_2}{\lambda_2} + \cdots\right)\lambda_1\lambda_2 \\ &= det\, g\left[1 + \sum\frac{\delta\lambda_i}{\lambda_i}\right] \\ &= det\, g + \sum\frac{\delta\lambda_i}{\lambda_i}det\, g \end{aligned}$$

Now,

$$\therefore \delta\sqrt{-g} = \delta\sqrt{-det\, g_{\alpha\beta}} = \frac{1}{2}\sqrt{-g}\, g^{\alpha\beta}\partial g_{\alpha\beta}$$

$$\delta g^{\alpha\beta} = -g^{\alpha i}g^{\beta i}\delta g_{ij}$$

$$\delta S = \frac{-T}{2}\int d^2\sigma\left\{\left[\frac{1}{2}\sqrt{-g}\, g^{ij}\delta g_{ij}\right]g^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x^\nu\eta_{\mu\nu} + \sqrt{-g}\left[-g^{\alpha i}g^{\beta j}\delta g_{ij}\right]\partial_\alpha x^\mu\partial_\beta x^\nu\eta_{\mu\nu}\right\}$$

$$-\frac{T}{2}\int d^2\sigma\sqrt{-g}\left\{\frac{1}{2}g^{ij}g^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x^\nu - \partial^i x^\mu\partial^j x^\nu\right\}\delta g_{ij}\eta_{\mu\nu}$$

Since,

$$\delta S = 0 \qquad \forall\ \delta g_{ij}$$

$$\therefore \frac{1}{2}g^{ij}g^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x^\nu\eta_{\mu\nu} = \partial^i x^\mu\partial^j x^\nu\eta_{\mu\nu}$$

let, $g^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu} = f^{-1}$

$$\implies g^{ij} = 2f\partial^i x^\mu \partial^j x^\nu \eta_{\mu\nu}$$

$$\implies g_{ij} = 2f\partial_i x^\mu \partial_j x^\nu \eta_{\mu\nu}$$

2.2.5 Conformal Field Theory

In the section 2.1 we have discussed a detailed version of Conformal field theory, well in this section we will put some light on the concept of holomorphic and anit-holomorphic functions.

A conformal transformation is a change of coordinates, [2]

$$\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma)$$

such that the metric transforms as,

$$g_{\alpha\beta}(\sigma) \to \Omega^2(\sigma) g_{\alpha\beta}(\sigma)$$

When the metric is dynamical, the transformation happens to be diffeomorphism. This is known as *Gauge Symmetries* [2].

The Euclidean worldsheet coordinates are $(\sigma^1, \sigma^2) = (\sigma^1, i\sigma^2)$

$$z = \sigma^1 + i\sigma^2 \quad ; \quad \tilde{z} = \sigma^1 - i\sigma^2$$

By this analogy, we refer the above functions to be as follows [2]:

- Holomorphic functions → "left-moving"
- Anit-Holomorphic functions → " right-moving"

The holomorphic derivative are [2],

$$\partial_z = \partial = \frac{1}{2}(\partial_1 - i\partial_2)$$

and

$$\partial_{\bar{z}} = \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2)$$

and $\partial_z = \bar{\partial z} = 1$ and $\partial \bar{z} = \bar{\partial}_z = 0$ and the metric is Euclidean as,

$$ds^2 = (d\sigma^1)^2 + (d\sigma^2)^2 = dzd\bar{z}$$

Hence, the components will be,

$g_{zz} = g_{\bar{z}\bar{z}} = 0$ and $g_{z\bar{z}} = \frac{1}{2}$

the measure factor is $dzd\bar{z} = 2d\sigma^1 d\sigma^2$. The delta function is defined as,

$$\int d^2 z \delta(z, \bar{z}) = 1$$

this springs up the factor of 2 difference between two delta functions [2].

$$v^z = (v^1 + iv^2) \quad ; \quad v^{\bar{z}} = (v^1 - iv^2)$$

and

$$v_z = (v^1 - iv^2) \quad ; \quad v_{\bar{z}} = (v^1 + iv^2)$$

The Holomorphy of the conformal transformations

Holomorphy is study of holomorphic function. As holomorphic functions are analytic functions in complex analysis.

Now consider,

$z = z' = f(z)$ and $\bar{z} = \bar{z}' = f(\bar{z})$

$$ds^2 = dzd\bar{z} \to \left|\frac{df}{dz}\right|^2 dzd\bar{z}$$

The space of conformal transformation is a finite dimension group, in the higher dimensions. For the theories which are defined on $\mathbb{R}^{p,q}$, conformal group is $SO(p+1, q+1)$ and $p+q > 2$ [2].

Some examples of 2d conformal transformation are [2]:

- $z \to z + a$: This is a transformation.
- $z \to \xi z$: This is a notation for $|\xi| = 1$ and a scale transformation (also known as dilation) for real $\xi \neq 1$

Consider an Energy-momentum tensor ($T_{\alpha\beta}$) as,

$$T_{\alpha\beta} = -\frac{4\pi}{\sqrt{g}}\frac{\delta S}{\delta g^{\alpha\beta}} \implies T^{\alpha\beta} = \frac{4\pi}{\sqrt{g}}\frac{\delta S}{\delta g_{\alpha\beta}} \tag{2.36}$$

$$S = \frac{1}{4\pi\alpha}\int d^2\sigma\sqrt{g}\, g^{\alpha\beta}\partial_\alpha x \partial_\beta x \tag{2.37}$$

$$4\pi\delta S = \frac{1}{\alpha'}\int d^2\sigma\left\{\left(\frac{1}{2}\sqrt{g}\, g^{ij}\delta g_{ij}\right) g^{\alpha\beta}\partial_\alpha x\partial_\beta x + \sqrt{g}(-g^{\alpha i}g^{\beta j}\delta g_{ij})\partial_\alpha x\partial_\beta x\right\} \tag{2.38}$$

$$\frac{4\pi}{\sqrt{g}}\frac{\delta S}{\delta g_{ij}} = -\frac{1}{\alpha'}\left[-\frac{1}{2}g^{ij}g^{\alpha\beta}\partial_\alpha x\partial_\beta x + \partial^i x\partial^j x\right] = T^{ij} \tag{2.39}$$

Now,

$$T(z,\bar{z}) = T_{zz}(z,\bar{z})$$

$$T_{ij} = -\frac{1}{\alpha'}\left[\partial^i x\partial^j x - \frac{1}{2}g^{ij}g^{\alpha\beta}\partial_\alpha x\partial_\beta x\right]$$

$$T = T_{zz} = -\frac{1}{\alpha'}[\partial_z x \partial_z x - 0]$$

$$\bar{T} = T_{\bar{z}\bar{z}} = -\frac{1}{\alpha'}[\partial_{\bar{z}} x \partial_{\bar{z}} x - 0]$$

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \delta^{\alpha\beta} \partial_\alpha x \partial_\beta x \tag{2.40}$$

$$\delta_\alpha S = \frac{1}{4\pi\alpha'} \int d^2\sigma \delta^{\alpha\beta} \partial_\alpha x \partial_\beta x + \frac{1}{4\pi\alpha'} \int d^2\sigma \delta^{\alpha\beta} \partial_\alpha x \partial_\beta \delta x \tag{2.41}$$

$$= \frac{1}{2\pi\alpha'} \int d^2\sigma \delta^{\alpha\beta} \partial_\alpha \delta x \partial_\beta x \tag{2.42}$$

$$= \frac{1}{2\pi\alpha'} \int d^2\sigma \delta x \left[\partial_\alpha \partial_\beta x \delta^{\alpha\beta}\right] \tag{2.43}$$

$$\delta_\alpha S = 0 \qquad \forall\ \delta x \implies \partial_\alpha \partial_\beta x = 0$$

$$\partial_z \partial_z x = 0 \ \rightarrow \nabla^2 x = 0 \tag{2.44}$$

$$\partial_z \partial_{\bar{z}} x = 0 \tag{2.45}$$

$$\partial_{\bar{z}} \partial_{\bar{z}} x = 0 \rightarrow \partial_{\bar{z}} \partial_z \bar{x} = 0 \tag{2.46}$$

2.3 Quantum Aspects

Consider an action,

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g}\, g^{\alpha\beta} \partial_\alpha x \partial_\beta x \tag{2.47}$$

then, generating functional for any theory with a given action is given by,

$$Z = \int \mathcal{D}x e^{-S[x]}$$

1. By using the variational principle we get
   $\int \mathcal{D}x \frac{\delta}{\delta x(z,\bar{z})} e^{-S[x]} = 0$
   $\implies \int \mathcal{D}x \left\{ \frac{1}{4\pi\alpha'} \partial^\alpha \partial_\alpha x \right\} e^{-S} = \implies \langle \partial^\alpha \partial_\alpha \hat{x} \rangle = 0$

2. The expectation value of well behaved operator is given by
   $\langle \hat{o} \rangle = \int \mathcal{D}x \mathcal{O} e^{-S[x]}$

3. In general, the variation of the expectation value of n such operators is as follows:
   $\int \mathcal{D} \frac{\delta}{\delta x(z,\bar{z})} \left[\mathcal{O}_1(z_1, \bar{z_1}), \cdots, \mathcal{O}_n(z_n, \bar{z_n}) e^{-S[x]} \right] = 0$
   $\implies \int \mathcal{D}x\, \mathcal{O}_1, \cdots, \mathcal{O}_n \frac{\delta}{\delta x} e^{-S} = 0$
   $\implies \int \mathcal{D}x\, [\mathcal{O}_1, \cdots, \mathcal{O}_n \partial^\alpha \partial_\alpha x]\, e^{-S} = 0$
   $\implies \langle \mathcal{O}_1, \cdots, \mathcal{O}_n \partial^\alpha \partial_\alpha \hat{x} \rangle = 0$
   and in here $\mathcal{O}_\rangle$ are arbitrary which $\boxed{\implies \partial^\alpha \partial_\alpha \hat{x} = 0}$

Now, we can insert a specific action given by:

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g}\, g^{\alpha\beta} \partial_\alpha x \partial_\beta x$$

we get from the above formula

$$\int \mathcal{D} \frac{\delta}{\delta x^\mu(z,\bar{z})} \left[\mathcal{O}_1, \cdots, \mathcal{O}_n x^\nu(z,\bar{z}) e^{-S[x]} \right] = 0$$

$$\Longrightarrow \int \mathcal{D}x \left\{ \mathcal{O}_1, \cdots, \mathcal{O}_n \left[\delta^\nu_\mu \delta_{(z)}(z - z') + x^\nu(z' - \bar{z}') \frac{1}{2\pi\alpha'} \partial^\alpha \partial_\alpha x^\mu(z, \bar{z}) \right] \right\} e^{-S} = 0$$

$$\langle \delta^\nu_\mu \delta_{(z)} \rangle + \frac{1}{2\pi\alpha'} \langle x^\nu(z', \bar{z'}) \partial^\alpha \partial_\alpha x_\mu \rangle = 0$$

$$\langle x^\nu(z', \bar{z'}) \partial^\alpha \partial_\alpha x^\mu(z, \bar{z}) \rangle = \frac{-1}{4\pi\alpha'} \eta^{\mu\nu} \delta_{(z)}(z - z')$$

$$\partial^\alpha \partial_\alpha \langle xx \rangle = -\frac{1}{2\pi\alpha'} \eta^{\mu\nu} \delta_{(z)}$$

but by using the formula we get our final equation as $\partial_z \partial_{\bar{z}} \ln |z - z'|^2 = 2\pi \delta_{(z)}(z - z')$

$$\Longrightarrow \langle x^\mu(z, \bar{z}) x^\nu(z', \bar{z'}) \rangle = -\frac{\alpha}{2} \ln |z - z'|^2$$

$$\hat{x}^\mu(z, \bar{z})) \hat{x}^\mu(z', \bar{z'}) = \frac{-\alpha'}{2} \ln |z - \bar{z'}|^2$$

Operator Product Expansion (OPE)

In order to study the locality, we need to consider the product of the operator at different points and see their behavior. In this terms locality is a statement about what happens as local operator approach each others.

$\mathcal{O}_i \to$ all local operator of the CFT.

$$\mathcal{O}_i(z, \bar{z}) \mathcal{O}_j(w, \bar{w}) = \sum_k c^k_{ij}(z - w, \bar{z} - \bar{w}) \mathcal{O}_k(w, \bar{w})$$

Let us see, how?

$$z = \int \mathcal{D}\phi e^{-S[\phi]}$$

ϕ denotes all the fields. Symmetry in QFT is such as an infinitesimal transformation,

$$\phi' = \phi + e\delta\phi$$

the measure and the action both are left invariant,

$$S[\phi'] = S[\phi]$$

and

$$\mathcal{D}\phi' = \mathcal{D}\phi$$

$$\epsilon \to \epsilon(\rho)$$

because of the infinitesimal transformation, both action and measure are invariant but, to leading of ϵ. The change has to be proportional to $\partial\epsilon$. We have,

$$\begin{aligned} z &= \int \mathcal{D}\phi' \exp\{-S[\phi']\} \\ &= \int \mathcal{D}\phi' \exp\left\{-S[\phi'] - \frac{1}{2\pi}\int J^{\alpha}\partial_{\alpha}\epsilon\right\} \\ &= \int \mathcal{D}\phi e^{-S[\phi]}\left(1 - \frac{1}{2\pi}\int J^{\alpha}\partial_{\alpha}\epsilon\right) \end{aligned}$$

where, $\int \to \int d^2\sqrt{g}$. The integrand has changed, value of partition functions can't have changed. We defined a dummy integration variable ϕ. Therefore the expression must be equal to original Z [2].

$$\int \mathcal{D}\phi e^{-S[\phi]}\left(\int J^{\alpha}\partial_{\alpha}\epsilon\right) = 0 \tag{2.48}$$

Equation 2.48 is the Noether theorem (quantum mechanically), the vacuum expectation value of the divergence of the current vanished.

$$\langle \partial_{\alpha} J^{\alpha} \rangle = 0$$

The time ordering correlation function can be described by,

$$\langle \mathcal{O}_1(\sigma_1)\cdots \mathcal{O}_n(\sigma_n)\rangle = \frac{1}{z}\int \mathcal{D}\phi e^{-S[\phi]}\mathcal{O}_1(\sigma_1)\cdots \mathcal{O}_n(\sigma_n) \tag{2.49}$$

2.4 Ward Identity for Conformal Transformation

Ward identities hold for the symmetries. In this section we will look what happens when we apply the ward identities to conformal transformation. Ward indentites are important aspects in quantum field theory and since, they are deeply related to the unitarity of the theory. Which in turn is a necessary condition for, being a physical quantum field theory. Let us briefly consider Ward Identity in our present context. We are working in two dimension and line integral around the boundary. Let $\hat{n}^\alpha$ - unit vector which is normal to boundary. Then, for any J^α (vector).

$$\int_\epsilon \partial_\alpha J^\alpha = \oint_{\partial\epsilon} J_\alpha \hat{n}^\alpha = \oint_{\partial\epsilon} (J_1 d\sigma^2 - J_2 d\sigma^1) \tag{2.50}$$

$$= -i\oint_{\partial\epsilon} (J_z dz - J_{\bar{z}} d\bar{z}) \tag{2.51}$$

$\sigma^\alpha \rightarrow$ Cartesian coordinates.

$$J_z = \frac{1}{2}(J_1 - iJ_2) \tag{2.52}$$

$$J_{\bar{z}} = \frac{1}{2}(J_1 + iJ_2) \tag{2.53}$$

Applying this to Ward Identity;

$$-\frac{1}{2\pi}\int \partial_\alpha \langle J^\alpha(\sigma)\mathcal{O}_1(\sigma_1)\cdots\rangle = \langle \delta\mathcal{O}_1(\sigma_1)\cdots\rangle = \langle \delta\mathcal{O}_1(\sigma_1)\cdots\rangle \tag{2.54}$$

Hence, the 2-D theory as,

$$\frac{i}{2\pi}\int dz\langle J_z(z,\bar{z})\mathcal{O}_1(\sigma_1)\cdots\rangle - \frac{i}{2\pi}\oint_{\partial\epsilon} d\bar{z}\langle J_z(z,\bar{z})\mathcal{O}_1(\sigma_1)\cdots\rangle = \langle\mathcal{O}_1(\sigma_1\rangle \tag{2.55}$$

As J_z is holomorphic while J_z is anti-holomorphic, hence contour integral picks up the residue. This can be understood by,

$$\frac{i}{2\pi}\oint_{\partial\epsilon} dz J_z(z)\mathcal{O}_1(\sigma_1) = -\operatorname{Res}[J_z\mathcal{O}_1] \tag{2.56}$$

$\delta z = \epsilon(z)$, we get;

$$\delta\mathcal{O}_1(\sigma_1) = -\operatorname{Res}[J_z(z)\mathcal{O}_1(\sigma_1)] = -\operatorname{Res}[\epsilon(z)T(z)\mathcal{O}_1(\sigma_1)] \tag{2.57}$$

In equation 2.57, $-\operatorname{Res}[\epsilon(z)T(z)\mathcal{O}_1(\sigma_1)]$, we have

$$\delta\mathcal{O}_1(\sigma_1) = -\operatorname{Res}\left[\bar{J}_{\bar{z}}(\bar{z})\mathcal{O}_1(\sigma_1)\right] = \operatorname{Res}\left[\bar{\epsilon}(\bar{z})\bar{T}(\bar{z})\mathcal{O}_1(\sigma_1)\right] \tag{2.58}$$

In the equation 2.58, the minus $(-)$ sign signifies the fact $\oint d\bar{z}$ i.e., the boundary integral is considers to be in the opposite direction.

The equation 2.58, is a way to know the the transformation happening in the operator under the conformal symmetry, provided we have knowledge of the OPE underlying between the stress-energy tensor i.e., $T(z)$ and $\bar{T}(\bar{z})$ and the operator.

3 Discussion

3.1 Primary operators

Consider an example

$\delta z = \epsilon(\text{constant})$

Operator transformation follows as,

$$\mathcal{O}(z-\epsilon) = \mathcal{O}(z) - \epsilon\partial\mathcal{O}(z) + \cdots$$

Noether current's translation is Stress- Energy tensor.

According to Ward Identities, we understand the OPE of T with any operator $\mathcal{O}$ has to be of the form,

$$T(z)\mathcal{O}(w,\bar{w}) = \cdots + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + \cdots$$

Operator with $\bar{T}$ will be,

$$\bar{T}(\bar{z})\mathcal{O}(w,\bar{w}) = \cdots + \frac{\bar{\partial}\mathcal{O}(w,\bar{w})}{\bar{z}-\bar{w}} + \cdots$$

Definition: *An operator $\mathcal{O}$ is said to have weight $(h,\tilde{h})$ is, under $\delta z = \epsilon z$ and $\delta\bar{z} = \bar{\epsilon}\bar{z}$, $\mathcal{O}$ transforms as [2],*

$$\delta\mathcal{O} = -\epsilon(h\mathcal{O} + z\partial\mathcal{O}) - \bar{\epsilon}(\tilde{h}\mathcal{O} + \bar{z}\bar{\partial}\mathcal{O}) \tag{3.1}$$

for operator $\mathcal{O}$ the weight $(h,\tilde{h})$, the OPE with T and $\bar{T}$ takes the form,

$$T(z)\mathcal{O}(w,\bar{w}) = \cdots + \frac{h\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{(z-w)} + \cdots \tag{3.2}$$

$$\bar{T}(\bar{z})\mathcal{O}(w,\bar{w}) = \cdots + \frac{\tilde{h}\mathcal{O}(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}(w,\bar{w})}{\bar{z}-\bar{w}} + \cdots \tag{3.3}$$

We will understand with the help of couple example, how can we conclusively say if an operator is a primary operator or not. Consider an example as follows:

EXAMPLE: 1

$$x(z)z(w) = \frac{-\alpha}{2}\ln(z-w)$$

$$\partial_z x(z)x(w) = \frac{-\alpha}{2}\frac{1}{z-w}$$

$$\partial_z x(z)\partial_w x(w) = \frac{-\alpha}{2}\frac{1}{(z-w)^2}$$

$$\partial x(z)\partial x(w) = \frac{\alpha'}{2} + \frac{1}{z-w}$$

$$T(z)\mathcal{O}(w) = \frac{h\mathcal{O}(w)}{(z-w)^2} + \frac{\partial\mathcal{O}(w)}{(z-w)} + \cdots$$

Now,

$$T(z)\partial_w x(w) = \frac{-1}{\alpha} : \partial x(z)\partial x(z)\partial x(w) : -\frac{1}{\alpha} : \partial x(z)\partial x(z)\partial x(w) : \tag{3.4}$$

$$= \frac{1}{(z-w)^2}\partial x(z) \tag{3.5}$$

By using Taylor expansion,

$$T(z)\partial_w x(w) = \frac{\partial x(w)}{(z-w)^2} + \frac{\partial(\partial x(w))}{(z-w)} + finite\ terms \tag{3.6}$$

Now, by comparing the weight h of an operator $\mathcal{O}(w) = \partial_w x(w)$ is 1.

EXAMPLE: 2

$$\begin{aligned}
T(z)x(w) &= -\frac{1}{\alpha} : \partial x(z)\partial x(z) : \alpha(w) \\
&= -\frac{1}{\alpha} : \partial x(z)\partial x(z)x(w) + \partial x(z)\partial x(z)x(w) : \\
&= -\frac{2}{\alpha}\left(-\frac{\alpha}{2}\frac{1}{z-w}\partial x(z)\right) \\
&= \frac{1}{z-w}\partial x(z)
\end{aligned}$$

Similarly by using Taylor Expansion,

$$T(z)x(w) = \frac{1}{z-w}\left\{\partial x(w) + (z-w)\partial(\partial x(w))\right\} \tag{3.7}$$

$$T(z)x(w) = \frac{1}{z-w}\partial x(w) + finite\ terms \tag{3.8}$$

The above given equation 3.8 doesn't composes with the equation of primary operator. hence, $T(z)x(w)$ cannot be considered as a primary operator and also $h \neq 0$.

EXAMPLE: 3

$\mathcal{O}(w) = e^{ikx}$, compute the OPE.

$T(z) = -\frac{1}{\alpha} : \partial x \partial x :$

as we know

$$x(z)x(\bar{z'}) = -\frac{\alpha'}{2}\ln(z-z') + constant$$

$$(z-z')^2 = (z_1 - z_1')^2 + (z_2 - z_2')^2 \tag{3.9}$$

$$= (z-w)(z-w) \tag{3.10}$$

$$\therefore x(z)(z') = -\frac{\alpha}{2} - \frac{\alpha}{2}\ln(\bar{z}-\bar{w}) + constant \tag{3.11}$$

Hence,

$$\partial x(z) z(w) = -\frac{\alpha'}{2} + non - singular \tag{3.12}$$

Now, we compute,

$$\begin{aligned}
\partial x(z) : e^{ikx(w)} : &= \sum_{m=0}^{\infty} \frac{(ik)^m}{m!} \partial x(z) : x(w)^m : && (3.13)\\
&= \sum_{m=1}^{\infty} \frac{(ik)^m}{m!} \partial x(z) : \partial x(w)^m : \partial x(z) && (3.14)\\
&= \sum_{m=1}^{\infty} \frac{(ik)^m}{m!} \partial x(z) x(w) : \partial x(w)^{m-1} : m + \partial x(z) && (3.15)\\
&= \sum_{m=1}^{\infty} \frac{(ik)^m}{m!} \left(\frac{-\alpha'}{2}\right) \frac{1}{z-w} : \partial x(w)^{m-1} : m + nonsingular && (3.16)\\
&= \frac{-i\alpha' k}{2} \frac{1}{z-w} \sum_{m=1}^{\infty} \frac{1}{(m-1)!} (ik)^{m-1} : x(w)^{m-1} : + nonsingular && (3.17)
\end{aligned}$$

Now, we compute,

$$\begin{aligned}
T(z) : e^{ikx(w)} : &= -\frac{1}{\alpha'} : \partial x(z) \partial x(z) :: e^{ikx(w)} : && (3.18)\\
&= -\frac{2}{\alpha'} : \partial x(z) \partial x(z) e^{ikx(w)} : && (3.19)\\
&= -\frac{1}{\alpha'} : \partial x(z) \partial x(z) \sum_{m=0}^{\infty} \frac{(ik)^m}{m!} x(w)^m : && (3.20)
\end{aligned}$$

1^{st} term is because of 1 contraction, while the 2^{nd} is because of 2 contraction.

$$T(z) : e^{ikx(w)} := ik : \frac{\partial x(z) e^{ikx(w)}}{(z-w)} : -\frac{1}{\alpha'} \frac{\alpha'^2}{4} \frac{1}{(z-w)^2} (ik)^2 : e^{ikx(w)} : + nonsingular \tag{3.21}$$

So, we have,

$$T(z):e^{ikx(w)}:=\frac{\alpha' k^2}{4}:\frac{e^{ikx(w)}}{(z-w)^2}:+(ik):\frac{\partial x(z)e^{ikx(w)}}{(z-w)}:+nonsingular \tag{3.22}$$

But, $\partial x(z)=\partial x(w)+(z-w)\partial^2 x(w)$,

So,

$$T(z):e^{ikx(w)}:=\frac{\alpha' k^2}{4}:\frac{e^{ikx(w)}}{(z-w)^2}:+(ik):\frac{\partial x(z)e^{ikx(w)}}{(z-w)}:+ik:\partial^2 x(w)e^{ikx}:+nonsingular \tag{3.23}$$

$\therefore :e^{ikx}:$ is a primary operator with weight, $h=\frac{\alpha' k^2}{4}$.

Note: L_0(defined inn section 3.1.1) eigenvalues is weight (h) of the operators.

C is for Casimir

T under the finite conformal transformation $z \to \tilde{z}(z)$ is,

$$\tilde{T}(\tilde{z})=\left(\frac{\partial \tilde{z}}{\partial z}\right)^2\left[T(z)-\frac{c}{12}S(\tilde{z},z)\right]$$

The T in the equation is independent of itself. It will be same on all state, the only affecting terms will be the constant terms or zero mode in the energy. In other words we can say it is the Casimir energy of the system. The S in the equation denotes the Schwarzian which is defined as

$$S(z,\tilde{z})=\left(\frac{\partial^3 \tilde{z}}{\partial z^3}\right)\left(\frac{\partial \tilde{z}}{\partial z}\right)^{-1}-\frac{3}{2}\left(\frac{\partial^2 \tilde{z}}{\partial z^2}\right)^2\left(\frac{\partial \tilde{z}}{\partial z}\right)^{-2}$$

Schwarzian has a tendency to preserve the group structure of successive conformal transformation. Let us, consider Euclidean cylinder, parametrised by

$$w=\sigma+i\tau \quad ; \quad \sigma\in[0,2\pi)$$

Creating a conformal transformation from the cylinder to the complex plane by [2],

$$z = e^{-iw}$$

There is a conformal mapping between cylinder and plane, hence if one understands the CFT on cylinder , one can immediately understand i on plane.

The constant time slices on cylinder are mapped on the plane. $z \neq 0$ (origin), $\tau \to -\infty$ (distant past).

Considering a Schwarzian $S(z,w) = \frac{1}{2}$

$$T_{cylinder}(w) = -z^2 T_{plane}(z) + \frac{c}{24}$$

Let ground state energy $= 0$ on plane $\to \langle T_{plane} \rangle = 0$.

3.1.1 The Virasoro Algebra

Radial Quantisation

$w \to$ complex coordinates on the cylinder.

$z \to$ coordinates on the plane.

Now,

$$w = \sigma + i\tau \quad , \quad z = e^{-iw}$$

States are sustained on the spatial space of $\sigma = constant$ and evolved by the Hamiltonian

$$H = \partial_\tau$$

Hamiltonian becomes dilation operator as mapped on plane.

$$D = z\partial + \bar{z}\bar{\partial}$$

Virasoro Generators

We will observe what happens to the stress tensor $T(z)$ evaluated on the plane. On, cylinder we expand T with the help of Fourier expansion.

Stress tensor $T(z)$,

$$T_{cylinder}(w) = -\sum_{m=\infty}^{\infty} L_m e^{imw} + \frac{c}{24}$$

After the Schwarzian transformation to the plane, $T_{cylinder}$ becomes Laurant expansion,

$$T(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}}$$

Taking a suitable contour integral,

$$L_n = \frac{1}{2\pi i} \oint dz\, z^{n+1}\, T(z) \tag{3.24}$$

$$\tilde{L_n} = \frac{1}{2\pi i} \oint d\bar{z}\, \bar{z}^{n+1}\, \bar{T}(\bar{z}) \tag{3.25}$$

Conformal transformation $\delta z = z^{n+1}$

$$J(z) = z^{n+1} T(z)$$

In quantum theory, conserved charge becomes generators for the transformation. Thus, L_n *and* $\tilde{L_n}$ *generators* $\delta_z = z^{n+1}$ *and* $\delta_{\bar{z}} = \bar{z}^{n+1}$. *They are known as the Virasoro generators [2]*

- L_{-1} *and* $\tilde{L_{-1}}$ *generate translations in the plane.*
- L_0 *and* $\tilde{L_0}$ *generate scaling and rotations.*

$$D = L_0 + \tilde{L_0}$$

$L_m \rightarrow$ contour integral over $\oint dz$

$L_n \rightarrow$ as a contour integral over $\oint dw$.

$$[L_m, L_n] = \left(\oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} - \oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i} \right) z^{m+1} w^{n+1} T(z) T(w) \tag{3.26}$$

$$\begin{aligned}[L_m, L_n] &= \oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i} z^{m+1} w^{n+1} T(z) T(w) \\ &= \oint \frac{dw}{2\pi i} \operatorname{Res} \left[z^{m+1} w^{n+1} \left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \cdots \right) \right]\end{aligned}$$

Computing Residue, $z = w$ Taylor expand z^{m+1} at w.

$$z^{m+1} = w^{m+1} + (m+1) w^m (z-m) + \frac{1}{2} m(m+1) w^{m-1} (z-w)^2 + \frac{1}{6} m(m^2-1) w^{m-2} (z-w)^3 + \cdots$$

and

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m(m^2-1) \delta_{m+n,0}$$

This is Virasoro Algebra.

In bosonic string theory, D = 26 dimensional Minkowski space.

$|\psi\rangle \rightarrow$ physical states subject to Virasoro coordinates.

$$L_n |\psi\rangle = 0 \qquad for\ n > 0$$

$$L_0 |\psi\rangle = h |\psi\rangle$$

where h is the weight.

Similarly for $\tilde{L_n}$:

$$\tilde{L_n} |\psi\rangle = 0 \qquad forn > 0$$
$$\tilde{L_0} |\psi\rangle = \tilde{h} |\psi\rangle$$

$(h, \tilde{h}) \rightarrow$ primary states of weight.

Conformal gauge is used to change coordinates to transform metric from $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ to $ds^2 = -e^{2\omega}dx^+dx^-$ Operator insertion in conformal gauge is

$$V \sim \int d^2z\mathcal{O} \tag{3.27}$$

Equation 3.27 the fact that we have dropped an over all normalisation constant.

Integrating over the world-sheet solves the problem of diffeomorphism.

The physical states are primary states of the CFT with weight $(+1, +1)$. Such operators are called VERTEX OPERATORS.

Vertex operator associated with the ground states of the string is,

$$V_{tachyon} \sim \int d^2z : e^{ipx} : \tag{3.28}$$

and $h = \tilde{h} = \frac{\alpha' p^2}{4}$ which we have already seen in example 3 3.1 in the section 2.3.

Weight $(+1, +1)$ possible only if the mass of the state is,

$$M^2 \equiv -p^2 = \frac{-4}{\alpha'} \tag{3.29}$$

On Euclidean cylinder the mode expansion is

$$X(w, \bar{w}) = x + \alpha' p\tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}(\alpha_n e^{inw} + \bar{\alpha}_n e^{in\bar{w}})$$

it is necessary to have some reality in Minkowski space hence , $\alpha_n^* = \alpha_{-n}$and $\tilde{\alpha}_n^* = \tilde{\alpha}_{-n}$.

Equation 3.29 represents, Mass of tachyon!

Looking at the first excited state of the form,

$$\xi_{\mu\nu}\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}\left|0;p\right\rangle$$

Where, $\xi_{\mu\nu}$ is constant term (to determine the type of state).

traceless symmetric $\xi_{\mu\nu} \rightarrow$ graviton

traceless anti-symmetric $\xi_{\mu\nu} \rightarrow B_{\mu\nu}$ field.

trace of $\xi_{\mu\nu} \rightarrow$ scalar for dilation.

$$L_n = \frac{1}{2\pi i}\oint dz\, z^{n+1}T(z) \tag{3.30}$$

Inverted form ,

$$T(z) = \sum_n z^{n-2}L_n \tag{3.31}$$

$$[L_n, L_m] = (n-m)L_{m+n} + \frac{c}{12}n(n^2-1)\delta_{n,-m} \tag{3.32}$$

Mode Expansion

This section is been referred form the String Theory. Volume 1, Introduction to the Bosonic String by Joseph Polchinski for more reference about the book please see reference [7].

Referring to equation in the box 3 in section 2.3 is solved further to give us the equation 3.33

$$\partial x^{\mu}(z) = -i\left(\frac{\alpha'}{2}\right)^{\frac{1}{2}}\sum_{m=-\infty}^{\infty}\frac{\alpha^{\mu}}{z^{m+1}} \tag{3.33}$$

$$\bar{\partial} x^{\mu}(\bar{z}) = -i\left(\frac{\alpha'}{2}\right)^{\frac{1}{2}}\sum_{m=-\infty}^{\infty}\frac{\bar{\alpha}^{\mu}}{\bar{z}^{m+1}} \tag{3.34}$$

Equivalently,

$$\alpha^{\mu}_{m} = \left(\frac{\alpha'}{2}^{\frac{1}{2}}\right) \oint \frac{dz}{2\pi} z^{m} \partial x^{\mu}(z) \tag{3.35}$$

$$\tilde{\alpha}^{\mu}_{m} = \left(\frac{\alpha'}{2}^{\frac{1}{2}}\right) \oint \frac{d\bar{z}}{2\pi} \bar{z}^{m} \bar{\partial} x^{\mu}(\bar{z}) \tag{3.36}$$

$i\partial_a x^{\mu}/\alpha'$ is the Noether current for space-time translations.

$$p^{\mu} = \frac{1}{2\pi} \oint (dz j^{\mu} - d\bar{z}\tilde{j}^{\mu}) = \left(\frac{\alpha'}{2}^{\frac{1}{2}}\right) \alpha^{\mu}_{0} = \left(\frac{\alpha'}{2}^{\frac{1}{2}}\right) \tilde{\alpha}^{\mu}_{0} \tag{3.37}$$

Integrating equation 3.33

$$\therefore x^{\mu}(z, \bar{z}) = \mathrm{x}^{\mu} - i\frac{\alpha'}{2} p^{\mu} \ln |z|^{2} + i \left(\frac{\alpha'}{2}\right)^{\frac{1}{2}} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{1}{m} \left(\frac{\alpha^{\mu}_{m}}{z^{m}} + \frac{\tilde{\alpha}^{\mu}_{m}}{\bar{z}^{m}}\right) \tag{3.38}$$

Either from standard canonical commutation, or from the contour argument and the xx OPE, one derivative,

$$[\alpha^{\mu}_{m}, \alpha^{\nu}_{n}] = [\tilde{\alpha}^{\mu}_{m}, \tilde{\alpha}^{\nu}_{n}] = m\delta_{m,-n}\eta^{\mu\nu} \tag{3.39}$$

$$[\mathrm{x}^{\mu}, p^{\nu}] = i\eta^{\mu\nu} \tag{3.40}$$

$$L_{m} \sim \frac{1}{2} \sum_{n=-\infty}^{\infty} (\alpha^{\mu}_{m-n}\, \alpha_{\mu n}) + a^{x} \tag{3.41}$$

'$\sim$' implies that we have ignore ordering of operators. For $m = 0$

$$L_{0} = \frac{\alpha' p^{2}}{4} + \sum_{n=1}^{\infty} (\alpha^{\mu}_{-n}\, \alpha_{\mu n}) \tag{3.42}$$

Equation 3.42 can also be represented as in the form of Normal ordering,

$$L_{m} = \frac{1}{2} + \sum_{n=-\infty}^{\infty} : \alpha^{\mu}_{m-n}\, \alpha_{\mu n} : \tag{3.43}$$

Similarly,

$$x^{\mu}(z,\bar{z})x^{\nu}(z',\bar{z'}) =: x^{\mu}(z,\bar{z})x^{\nu}(z',\bar{z'}) : +\frac{\alpha'}{2}\eta_{\mu\nu}\left[-\ln|z|^2+\sum_{m=1}\frac{1}{m}\left(\frac{z'^m}{z^m}+\frac{\bar{z'}^m}{\bar{z}^m}\right)\right] \quad (3.44)$$

$$=: x^{\mu}(z',\bar{z'})x^{\nu}(z',\bar{z'}) : -\frac{-\alpha}{2}\ln|z-z'|^2 \quad (3.45)$$

3.2 Coherent States

Coherent states were introduced by 'Roy J. Glauber'. He attempted to look for the superposition of eigenstates, which looked like classical results but would actually be the quantum states. So, basically coherent states are the eigenvectors of the annihilation operators of the theory.

As we know about the uncertainty principle,

$$\Delta x = \frac{\lambda_{osc}}{\sqrt{2}} \quad (3.46)$$

$$\lambda_{osc} = \frac{\hbar}{M\omega_{osc}} \quad (3.47)$$

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2} \quad (3.48)$$

Except $x_0 = 0$ and $p_0 = 0$, the average momentum and average position the wave behaves like a classical particle.

Since, coherent states are basically eigenstates of annihilation operator. Mathematically speaking, coherent states can be defined as the annihilator operator having an unique eigenstate which is associated to the eigenvalue. [12]

$$A\left|\alpha\right\rangle = \alpha\left|\alpha\right\rangle, \qquad \left\langle\alpha|\alpha\right\rangle = 1 \quad (3.49)$$

where α is a complex number. For every α there exist a different coherent state.

The coherent states are the classical state of the harmonic oscillator. In quantum optics they describes the quantum states of a laser.

3.2.1 Coherent states in quantum mechanics

Considering the famous 1 - dimensional Schrödinger equation, i.e.,

$$i\hbar\frac{\partial\left|\psi\right\rangle}{\partial t}=-\frac{\hbar}{2m}\frac{\partial^2\left|\psi\right\rangle}{\partial x^2}+V\left|\psi\right\rangle \tag{3.50}$$

Now,

$$-\frac{\hbar}{2m}\frac{\partial^2\left|\psi\right\rangle}{\partial x^2}+V\left|\psi\right\rangle=E\left|\psi\right\rangle \tag{3.51}$$

where V is the potential energy.

$$\left[\frac{p^2}{2m}+V(x)\right]\left|\psi\right\rangle=E\left|\psi\right\rangle \tag{3.52}$$

where, $\frac{p^2}{2m}+V(x)=H$ i.e., Hamiltonian or total energy of the system and

$$H=\frac{1}{2m}\left[p^2+(m\omega x)^2\right]$$

Now,

wave function of the harmonic oscillator can be defined by the Ladder operators.

The ladder operators are basically, the raising and lowering operator or also called as creating and annihilating operators respectively. They are mathematically defined as,

$$a=\frac{1}{\sqrt{2\hbar m\omega}}(ip+m\omega x)\quad\rightarrow\quad \textit{lowering operator} \tag{3.53}$$

$$a^\dagger=\frac{1}{\sqrt{2\hbar m\omega}}(-ip+m\omega x)\quad\rightarrow\quad \textit{raising operator} \tag{3.54}$$

Coherent states are the eigenstates of the annihilation operators, a.

$$a\,|\lambda\rangle = \lambda\,|\lambda\rangle\,, \qquad with \qquad |\lambda\rangle = \exp\bigl(\lambda a^\dagger - \lambda^* a\bigr)\,|0\rangle\,,$$

which there have classical expectation values

$$\langle x(t)\rangle = \frac{1}{\sqrt{2}}(\lambda^8 e^{i\omega t} + \lambda e^{-i\omega t}), \qquad with \qquad \frac{d^2}{dt^2}\langle x(t)\rangle = -\omega^2\langle x(t)\rangle.$$

where λ is a quantum number.

3.3 Coherent states in String theory

3.3.1 States and Vertex Operator

$$L_n\,|\psi\rangle = 0 \quad \forall\, n > 0 \tag{3.55}$$

$$L_0\,|\psi\rangle = a\;|\psi\rangle \tag{3.56}$$

$|\psi\rangle \rightarrow$ physical states subject to Virasoro constraints.

Similarly, for $\tilde{L}_n$

$$\tilde{L_n}\,|\psi\rangle = 0 \qquad n > 0 \tag{3.57}$$

$$\tilde{L_0}\,|\psi\rangle = h\,|\psi\rangle \tag{3.58}$$

Since, $L_0 + \tilde{L_0}$ is Hamiltonian: The term "Highest weight" for a lower energy state [1]. L_n decreases the eigenvalue of L_0 by n [1],

$$L_0 L_n\,|\psi\rangle = (L_n L_0 - nL_n)\,|\psi\rangle = (h-n)L_n\,|\psi\rangle$$

if $L_0\,|\psi\rangle = h\,|\psi\rangle$

if $|h\rangle$ is a highest weight state, then $|\psi\rangle$ is annihilated by L_n with n¿0.

$$L_n |\psi\rangle = 0 \,, \qquad for\ n \geqslant 1$$

If L_0 acting on $|\psi\rangle$ (highest weight state) creates a state $|h\rangle'$ Then,

$$L_n |h\rangle' = 0 \qquad n \geqslant 1$$

3.3.2 DDF operators

The DDF in the DDF operators stands for "DelGuidice, DiVecchia and Fubini". DDF operators have special character, they commute with the Virasoro algebra. If you apply them to a physical state, that produces another physical state because as they commute with the Virasoro generators, so if a state $|\psi\rangle$ satifies the physical constraints given by the Virasoro operators, then so does $A^* |\psi\rangle$

DDF operators can be mathematically defined by,

$$A^i_n = \oint \frac{dz}{2\pi} V^i(nk_0, z) \tag{3.59}$$

where,

$$V^i(nk_0, z) = \partial x^i(z) e^{ink_0 x^+(z)} (2/\alpha')^{\frac{1}{2}} \tag{3.60}$$

DDF operators can also be used to satisfy the oscillator algebra and hence, it perfectly imbibes the information about the string, except for the world-sheet ground states themselves.

$$[A^i_m, A^j_n] = m\delta^{ij}\delta_{m-n}\frac{\alpha' k_0 p^+}{2} \tag{3.61}$$

DDF formalism provides a database which relates every light-cone gauge state to the corresponding covariant gauge vertex operator.

$$N = \sum_j n_j \qquad and \qquad \bar{N} = \sum_j \bar{n}_j \qquad with \qquad N = \bar{N}$$

a general light-cone gauge mass eigenstate, state is of the form,

$$|v\rangle_{lc} = \frac{1}{\sqrt{2p^+\nu_{d-1}}} c\,\xi_{ij\cdots,kl\cdots} \times \alpha^i_{-n_1}\alpha^j_{-n_2}\cdots\alpha^{-k}_{-n_1}\alpha^{-l}_{-n_2}\cdots\left|0,0;p^+,p^i\right\rangle \tag{3.62}$$

To every light-cone gauge state there corresponds the correctly normalised covariant vertex operator of momentum k.

$$V(z,\bar{z}) = \frac{g_c}{\sqrt{2p^+\nu_{d-1}}} c\,\xi_{ij\cdots,kl\cdots} \times A^i_{-n_1}A^j_{-n_2}\cdots A^{-k}_{-n_1}A^{-l}_{-n_2}\cdots e^{ipx(z,\bar{z})} \tag{3.63}$$

with the element dimensionless DDF operators $A^i_{-n_1}A^j_{-n_2}$ defined by,

$$\left.\begin{aligned} A^i_n &= \sqrt{\frac{2}{\alpha'}}\oint \bar{d}z\,\partial_z x^i(z)e^{inqx(z)} \\ \bar{A}^i_n &= \sqrt{\frac{2}{\alpha'}}\oint \bar{d}\bar{z}\,\partial_z x^i(\bar{z})e^{inqx(\bar{z})} \end{aligned}\right\} \tag{3.64}$$

To construct string coherent states one proceeds by construction of coherent states in the harmonic oscillator, where by coherent states are constructed by exponential of the creation operator,

$$e^{\frac{-|\lambda|^2}{2}}e^{\lambda a^\dagger}\,|0\rangle \tag{3.65}$$

with $a\,|0\rangle$ and $[a,a^\dagger] = 1$

QUESTION: `What does it mean by a quantum state with a classical representation? String states with a classical interpretation should posses classical expectation values (with small uncertainties modulo zero mode contribution) provided these are compatible with the symmetries of string theory. These classical expectation values should be non-trivially consistent with the classical equation of motion` and constraints.

General equation of DDF on Vacuum

$$A_n^i = \oint \frac{dz}{2\pi} \partial x^i \, e^{ik_0 n x^+(z)} \frac{2}{\alpha'}^{\frac{1}{2}} \tag{3.66}$$

$$= \frac{2}{\alpha'}^{\frac{1}{2}} \oint \frac{dz}{2\pi} \partial x^i \, e^{ik_0 n x^+(z)} \tag{3.67}$$

$$A_m^i : e^{ikx} : = \sqrt{\frac{2}{\alpha'}} \oint dz \partial x^i e^{iNqx} e^{ipx} \tag{3.68}$$

$$A_m^i : e^{ikx} := \sqrt{\frac{2}{\alpha'}} \oint \frac{dz}{iz} \left(p^i z^{-N} + \sum_{r=1}^{\infty} \frac{i}{(r-1)!} \partial^r x^i z^{r-N} \right) \sum_{m=0}^{\infty} z^m S_m(N_{q;0}) e^{(p-Nq)x(0)} \tag{3.69}$$

where $S_m(N_{q;0}) \equiv S_m(a_1, \cdots, a_m)$ and $a_s = \frac{-inqd_z^s x}{S!}$ and

$$S_m(a_1, \cdots, a_m) \equiv -i \oint_0 \bar{d}u u^{-m-1} \exp\left[\sum_{s=1}^{m} a_s u^s \right] \tag{3.70}$$

$$\equiv \sum_{k_1 + 2k_2 + \cdots + mk_m = m} \frac{a_1^{k_1}}{k!} \cdots \frac{a_m^{k_m}}{k!} \tag{3.71}$$

Using operator product, intergrands are brought close to vacuum and we get gravition, which is,

$$\xi_{ij} A_{-1}^i A_{-1}^j e^{ipx(z,\bar{z})} = \frac{2}{\alpha'} \xi_{ij} \oint \bar{d}w \partial_w x^i(w) e^{-iqx(w)} \oint_{\bar{z}} \bar{d}\bar{w} \partial_{\bar{w}} x^j(w) e^{-iqx(w)} e^{ipx(z,\bar{z})} \tag{3.72}$$

$$\cong \frac{2}{\alpha'} \xi_{ij} \left(\delta_\mu^i - \frac{\alpha'}{2} p^i q_\mu \right) \left(\delta_\nu^i - \frac{\alpha'}{2} p^j q_\nu \right) \partial x^\mu(z) \bar{\partial} x^\nu(\bar{z}) e^{i(p-q)x(z,\bar{z})} \tag{3.73}$$

$$\zeta_{\mu,\nu} = \xi_{i,j} \left(\delta_\mu^i - \frac{\alpha'}{2} p^i q_\mu \right) \left(\delta_\nu^i - \frac{\alpha'}{2} p^j q_\nu \right) \tag{3.74}$$

4 Conclusion

This dissertation outlines the mass eigenstates covariant normal ordered vertex operator and the covariant coherent states vertex operator. This construction was possible because of the us of the DDF operators. As the coherent states are potentially macroscopic and have a direct correspondence with the classical evolving string. Hence they can be recognised with the fundamental cosmic string.

The role of Schur polynomial is one of the most important concepts as it provide a crucial role in construction of the flat space vertex operator. The DDF operators construction gives us the ability to create string coherent state covariant vertex operators.

Due to lack of time the aim of the dissertation was not able to met. But I would like to work on it in future and as a matter of fact, an immediate application of coherent states vertex operator in cosmic string evolution is that it is a possible to search for the classical computations and the string theory predictions.

5 Appendix

5.1 Contour Integrals

In the mathematical sector of complex analysis, contour integration is used to certain integrals in the complex plane. Contour integrals includes methods as mentioned below:

- Direct integration of complex functions in the contour,
- Cauchy's integral formula,
- Residue theorem.

5.1.1 Cauchy's Theorem

Theorem 5.1 (Cauchy's Theorem) *Let C be a simple closed curve with a continuous turning tangent except possibly are a finite number of points (that is, we allow a finite number of corners, but otherwise the curve must be "smooth"). if $f(z)$ is analytic on the inside C, then [5]*

$$\boxed{\oint_C f(z)dz = 0} \tag{5.1}$$

Proof of Cauchy's theorem is as follows

Conditions to be considered:

- $f(z)$ is holomorphic on and everywhere inside C.
- C is a simple curve (doesn't cross itself)
- C has a finite number of corners.

Assuming $\frac{\partial f}{\partial z}$ is also continuous.

Now,

$$f(z) = u + iv \tag{5.2}$$

and

$$dz = dx + idy \tag{5.3}$$

So,

$$\begin{aligned} \oint_C f(z)dz &= \oint_C (u + iv)(dx + idy) \\ &= \oint_C (udx - vdy) + i\left[\oint_C (vdx + udy)\right] \end{aligned} \tag{5.4}$$

Using the idea of Green's theorem,

$$\oint (Pdx + Qdy) = \iint_C \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \tag{5.5}$$

where C is simple, P and Q are continuous and have continuous derivatives.

Therefore from equation 5.4, we consider

$$\oint_C (udx - vdy) = I_1 \tag{5.6}$$

and

$$\oint_C (vdx + udy) = I_2 \tag{5.7}$$

for equation**??**, $P = u$ and $Q = -v$,

$$\therefore I_1 = \iint_C \left(\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy \tag{5.8}$$

According to the Cauchy-Riemann relation, we know $f(z) = u + iv$ is holomorphic, hence

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{5.9}$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{5.10}$$

$$I_1 = 0$$

from equation 5.9, similarly we get $I_2 = 0$ from equation **??**, whilch implies

$$\boxed{\oint_C f(z)dz = 0}$$

5.1.2 Cauchy's Integral Formula

Theorem 5.2 (Cauchy's Integral Formula) *Let $f(z_0)$ be analytic on a simply connected domain D. Suppose that $z \in D$ and C is a simple closed curve in D that encloses z_0. Then, [6]*

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz \tag{5.11}$$

Proof: Let z_0 be a point on the complex plan inside the curve C. Define, $g(z) = \frac{f(z)}{z-z_0}$, and $f(z)$ is holomorphic and inside C. Even g(z) is holomorphic everywhere except $z - z_0$. Hence, we get

$$\oint_w g(z)dz = 0 \tag{5.12}$$

If $C(z_0, r)$ denote the radius (r) around z_0 of the circle C, for a significantly small $r > 0$ then,

$$\boxed{f(z_0) = \frac{1}{2\pi i} \oint_{C(z_0,r)} \frac{f(z)}{z - z_0} dz} \tag{5.13}$$

Physical Interpretation: Given value of f on C (boundary of the curve), can be used to find value of f anywhere inside the curve C by applying Cauchy's Integral formula. [6] [5]

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