Numerical Simulation of Fluid Flow Using the Lattice Boltzmann Method: A Study on Convergence and Stability

Abstract

The Lattice Boltzmann Method (LBM) is a versatile and efficient numerical approach for solving fluid dynamics problems, particularly the Navier-Stokes equations. This report explores the application of LBM to a two-dimensional (2D) fluid simulation, focusing on the convergence and stability of solutions for varying relaxation times (τ). The simulation captures fluid dynamics with embedded density peaks and velocity distributions using the D2Q9 model, enhanced with non-linear equilibrium corrections. To assess stability, a convergence test was conducted across multiple τ values, revealing critical thresholds for numerical instability. The results provide insight into the delicate balance between computational accuracy and physical fidelity. This report includes visualizations such as time-evolving density maps, 3D scatter plots, and convergence behavior in logarithmic scale, making it accessible to a broad audience.

Keywords: Lattice Boltzmann Method, Computational Fluid Dynamics, Convergence Analysis, Numerical Stability, D2Q9 Model, Relaxation Time, Fluid Flow Simulation

1 Introduction

Numerical methods for solving partial differential equations (PDEs) are indispensable tools for modeling and understanding complex physical phenomena. Among these, the *Lattice Boltzmann Method (LBM)* has gained popularity due to its simplicity and effectiveness in simulating fluid flow problems [4].

LBM offers a mesoscopic perspective, bridging microscopic particle dynamics and macroscopic fluid behavior [3, 1]. The method evolves particle distribution functions on a lattice based on the Boltzmann transport equation:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f = \Omega, \tag{1}$$

where $f = f(\mathbf{x}, \mathbf{v}, t)$ is the particle distribution function, $\mathbf{v}$ is the particle velocity, and Ω represents the collision term. The seminal works of McNamara and Zanetti [3] laid the foundation for using the Boltzmann equation in lattice-gas automata, later formalized in LBM.

In this report, the D2Q9 lattice model is employed, where nine discrete velocity directions ($\mathbf{e}_i$) and corresponding weights (w_i) define the lattice [4]. The discrete lattice Boltzmann equation is expressed as:

$$f_i(\mathbf{x} + \mathbf{e}_i \Delta t, t + \Delta t) - f_i(\mathbf{x}, t) = -\frac{1}{\tau} \left[f_i(\mathbf{x}, t) - f_i^{\text{eq}}(\mathbf{x}, t) \right], \tag{2}$$

where f_i is the particle distribution along direction i, τ is the relaxation time, and f_i^{eq} is the equilibrium distribution function [1].

1.1 Equilibrium Distribution Function and Nonlinear Corrections

The equilibrium distribution function is fundamental to LBM and is written as [4]:

$$f_i^{\mathrm{eq}} = w_i \rho \left[1 + 3\mathbf{e}_i \cdot \mathbf{u} + \frac{9}{2}(\mathbf{e}_i \cdot \mathbf{u})^2 - \frac{3}{2}\mathbf{u}^2 \right], \tag{3}$$

where ρ is the macroscopic density, and $\mathbf{u} = (u_x, u_y)$ is the macroscopic velocity.

A non-linear correction to the equilibrium distribution was added in this study to enhance physical fidelity [1]:

$$f_i^{\mathrm{eq,\ corrected}} = f_i^{\mathrm{eq}} + \eta \rho u^2 (1 + \mathbf{e}_i \cdot \mathbf{u}), \tag{4}$$

where η is a correction factor. Such modifications are critical for capturing complex non-linear effects in fluid flows.

1.2 Macroscopic Variables and Stability

Macroscopic variables, such as density (ρ) and velocity ($\mathbf{u}$), are computed as moments of the distribution functions [2]:

$$\rho = \sum_{i=0}^{8} f_i, \quad \mathbf{u} = \frac{1}{\rho} \sum_{i=0}^{8} f_i \mathbf{e}_i. \tag{5}$$

The stability of the simulation is directly linked to the relaxation time (τ), which determines the fluid viscosity [1, 2]. When $\tau < 0.6$, simulations exhibit numerical instability due to error amplification.

2 Results and Discussion

Figure 1 shows snapshots of the density distribution at various time steps during the simulation. The time evolution captures the fluid flow dynamics, including the formation of density peaks and the development of velocity profiles. These snapshots provide a visual understanding of how the fluid evolves under the specified conditions. The simulation demonstrates the characteristic behavior of fluid flow, such as the spreading and mixing of density.

In Figure 2, we observe the convergence behavior of the simulation for various relaxation times (τ). The maximum density (ρ) is plotted against the number of simulation steps in logarithmic scale. The results show that for values of τ less than 0.6, the system becomes unstable, as indicated by the steep rise in density. This highlights the critical threshold at which numerical instability occurs, confirming that τ plays a crucial role in determining the stability of the LBM simulations.

Figure 3 shows a 3D scatter plot of the final density (ρ) distribution across the simulation domain. Each point represents the density value at a specific spatial coordinate. The color scale provides a clear visualization of density variations, allowing for an immediate assessment of fluid distribution. This figure highlights the areas of higher and lower density, showcasing how the fluid settles into equilibrium in the final stages of the simulation.

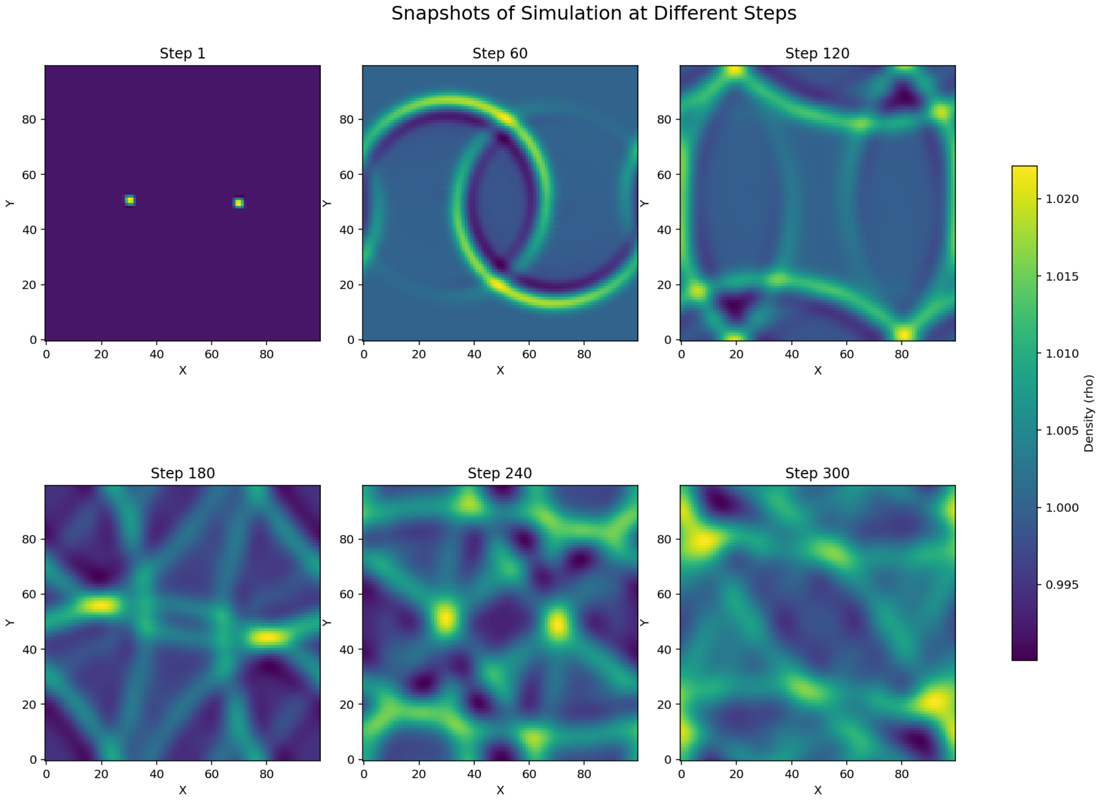

Figure 1: Snapshots of density distribution at selected simulation steps (3x2 grid). These highlight the evolution of fluid dynamics over time.

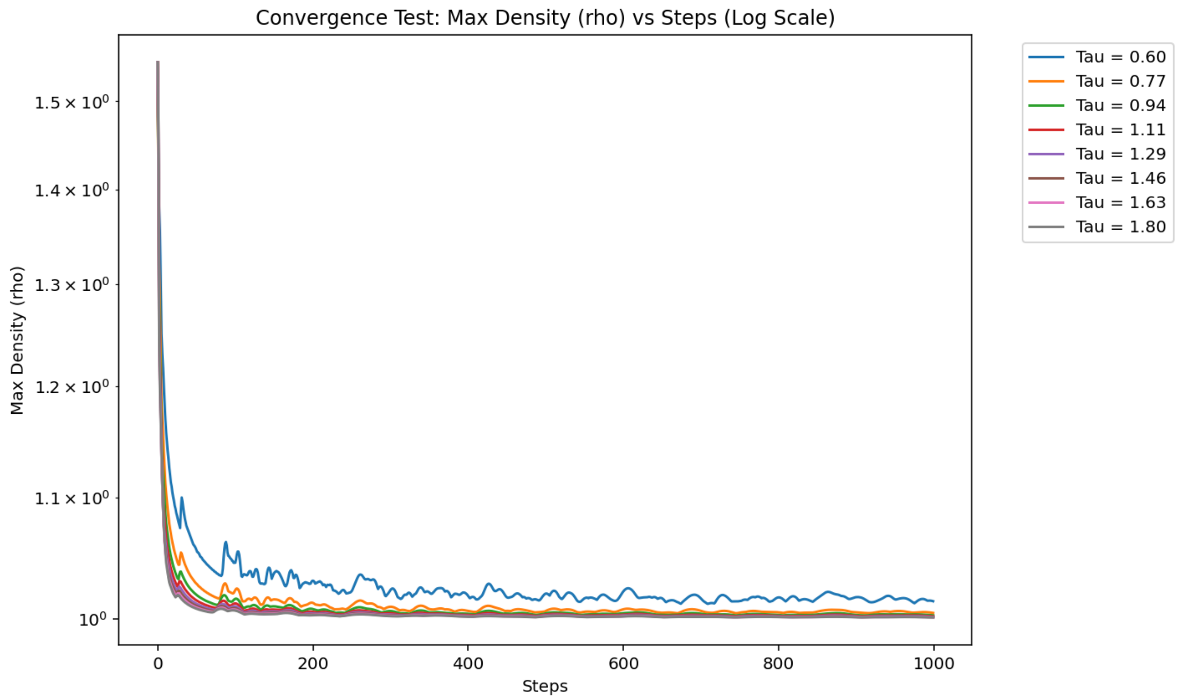

Figure 2: Convergence test: Maximum density (ρ) as a function of simulation steps for different τ values. The logarithmic scale highlights divergence at lower τ values.

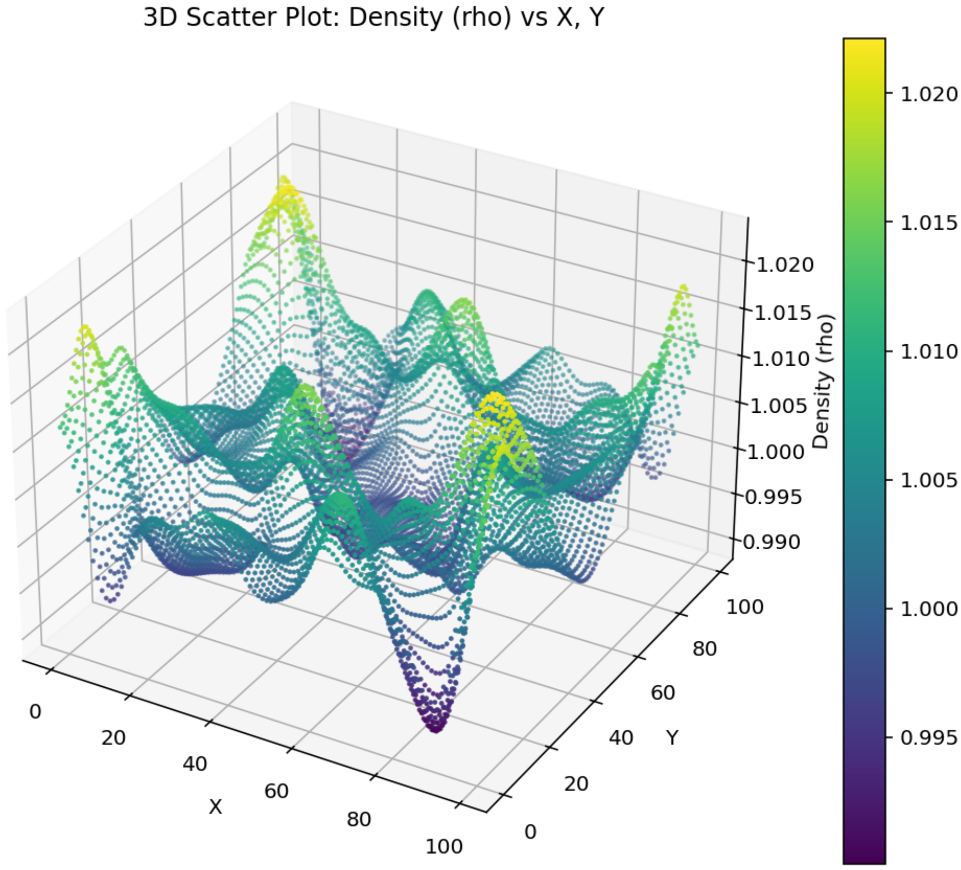

Figure 3: 3D scatter plot of the final density (ρ) against spatial coordinates. The color scale indicates density variations.

3 Conclusion

This study demonstrates the effectiveness of the Lattice Boltzmann Method (LBM) in solving 2D fluid dynamics problems, with a specific focus on the relationship between the relaxation time (τ) and numerical stability. By incorporating advanced visualizations and a convergence analysis, the report highlights key trade-offs in parameter selection and their impact on simulation fidelity.

Future work could involve extending this method to three-dimensional (3D) simulations, critical for modeling turbulence, multiphase flows, and aerodynamics. The method could also be integrated with thermal and reactive transport equations to study heat and mass transfer problems in engineering applications [4, 2]. Such advancements would make LBM highly relevant for microfluidics, biomedical devices, and climate modeling.

References

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